§ 11.1

#1. $\frac{dx_1}{dt} = 2x_1 + 3x_2$

$\frac{dx_2}{dt} = -4x_1 + x_2$

Sl: Let $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. then $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix} \vec{x}(t)$.

#20. $\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ $\quad x_1(0) = 2.\quad x_2(0) = -1$

Sl: $\lambda_1 = 1. \quad \lambda_2 = 2.$

$\therefore$ Eigenvector corresponding to $\lambda_1 = 1$ is:

$\begin{bmatrix} 1-1 & 3 \\ 0 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0} \Rightarrow \begin{cases} 3x_2 = 0 \\ x_2 = 0 \end{cases} \Rightarrow x_2 = 0$

$\therefore \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Eigenvector corresponding to $\lambda_2 = 2$ is:

$\begin{bmatrix} 1-2 & 3 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0} \Rightarrow \begin{cases} -x_1 + 3x_2 = 0 \\ 0 = 0 \end{cases} \Rightarrow x_1 = 3x_2$

$\therefore \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\therefore \vec{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + 3c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\therefore c_2 = -1$

$c_1 = 2 - 3c_2 = 5$

$\therefore \vec{x}(t) = 5e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

#24. $\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ $x_1(0) = 1$. $x_2(0) = 2$

Sl: $\det\left(\begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix} - \lambda I_2\right) = \det\begin{bmatrix} -3-\lambda & 4 \\ -1 & 2-\lambda \end{bmatrix} = (\lambda+3)(\lambda-2)+4 = \lambda^2+\lambda-2 = 0$

$(\lambda+2)(\lambda-1) = 0$ $\lambda_1 = -2$, $\lambda_2 = 1$.

Eigenvector corresponding to $\lambda_1 = -2$:

$\begin{bmatrix} -3+2 & 4 \\ -1 & 2+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$ $\begin{cases} -x_1 + 4x_2 = 0 \\ -x_1 + 4x_2 = 0 \end{cases} \Rightarrow x_1 = 4x_2$

$\therefore \vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Eigenvector corresponding to $\lambda_2 = 1$:

$\begin{bmatrix} -3-1 & 4 \\ -1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$ $\begin{cases} -4x_1 + 4x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \Rightarrow x_1 = x_2$

$\therefore \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x}(0) = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4c_1 + c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{cases} 4c_1 + c_2 = 1 & ① \\ c_1 + c_2 = 2 & ② \end{cases}$ ① − ② : $3c_1 = -1$ $c_1 = -\frac{1}{3}$. $c_2 = 2 - c_1 = \frac{7}{3}$

$\therefore \vec{x}(t) = -\frac{1}{3} e^{-2t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \frac{7}{3} e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\square$

#26. $\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ $x_1(0) = -3$. $x_2(0) = 1$

Sl: $\det\begin{bmatrix} 2-\lambda & 6 \\ 1 & 3-\lambda \end{bmatrix} = (\lambda-2)(\lambda-3)-6 = \lambda^2-5\lambda = 0 \quad \lambda_1=0. \quad \lambda_2=5.$

Eigenvector corresponding to $\lambda_1 = 0$: $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$ $\begin{cases} 2x_1+6x_2=0 \\ x_1+3x_2=0 \end{cases} \Rightarrow x_1+3x_2=0$

$\therefore \vec{v_1} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Eigenvector corresponding to $\lambda_2 = 5$: $\begin{bmatrix} 2-5 & 6 \\ 1 & 3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$ $\begin{cases} -3x_1+6x_2=0 \\ x_1-2x_2=0 \end{cases} \Rightarrow x_1 = 2x_2$

$\therefore \vec{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore \vec{x}(t) = c_1 e^{0t}\begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{x}(0) = \begin{bmatrix} -3c_1+2c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\therefore c_1 = 1. \quad c_2 = 0.$

$\therefore \vec{x}(t) = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ (a constant solution) $\square$

#27. Let $\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (*)$

(a) Show that $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has the repeated eigenvalues $\lambda_1 = \lambda_2 = 1$

(b) Show that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors of A and any vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ can be written as $c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) Show that $\vec{x}(t) = c_1 e^t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a solution of $(*)$ that satisfies the initial condition $x_1(0) = c_1, \quad x_2(0) = c_2$.

Pf: (a) $\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = 0 \quad \therefore \lambda_1 = \lambda_2 = 1.$

(b) $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors of A corresponding to the eigenvalue $\lambda = 1$.

and moreover $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) $\vec{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ c_2 e^t \end{bmatrix}$

$\therefore x_1(t) = c_1 e^t \qquad \frac{dx_1}{dt} = c_1 e^t = x_1(t)$

$x_2(t) = c_2 e^t \qquad \frac{dx_2}{dt} = c_2 e^t = x_2(t).$

$\therefore \frac{d\vec{x}}{dt} = \vec{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}(t)$

$\vec{x}(0) = \begin{bmatrix} c_1 e^0 \\ c_2 e^0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \therefore x_1(0) = c_1. \quad x_2(0) = c_2$ $\square$